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A class of generalized hypergeometric summations

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Abstract

Summations over the positive integers n of the generalized hypergeometric expressions $(\pm 1)^n {}_pF_{p+1}[-n^2x^2]$ $(x > 0)$ are derived in closed form. The specialization $p = 0$, for example, reduces to known results for Schlömilch series. In addition, we record the apparently not readily available sine and cosine transforms of ${}_pF_{p+1}[-b^2x^2]$ $(b > 0)$, the latter of which is used together with a form of the Poisson summation formula to deduce the aforementioned results.

Keywords: Series of generalized hypergeometric functions; Sine and cosine transforms

AMS classifications: 33C20; 44A20

1. Introduction

Recently, Miller deduced closed-form representations for two infinite hypergeometric sums which contain, for example, certain Schlömilch series, and summations of Lommel and Struve functions. In this note we shall derive generalizations of these results (cf. [2, Eqs. (4.5) and (4.6)]); that is, we shall deduce closed-form representations for

$$S(x) \equiv \sum_{n=1}^{\infty} (-1)^n {}_pF_{p+1}[(\alpha_p); (\beta_{p+1}); \; -n^2x^2] \tag{1.1a}$$

and

$$T(x) \equiv \sum_{n=1}^{\infty} {}_pF_{p+1}[(\alpha_p); (\beta_{p+1}); \; -n^2x^2], \tag{1.1b}$$

where p is a nonnegative integer. Convergence of these series will be discussed in Section 3. When $p = 0$, $S(x)$ and $T(x)$ are essentially sums of Bessel functions of the first kind and are referred to as Schlömilch series which are generalizations of Fourier series. When $p = 1$, by specialization of the parameters of ${}_1F_2[\alpha; \beta, \gamma; \; -n^2x^2]$, we obtain infinite sums of Lommel and Struve functions. All of the specializations just alluded to are recorded in [2].

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As before, we rely on a form of the Poisson summation formula [7, p. 60, Eq. (2.8.1)], namely

$$\sum_{n=0}^{\infty} f(n) = \frac{1}{2} f(0) + \int_0^{\infty} f(t)\,\mathrm{d}t + 2 \sum_{k=1}^{\infty} \int_0^{\infty} \cos(2\pi k t) f(t)\,\mathrm{d}t, \tag{1.2}$$

where the penultimate integral exists and $f(t)$ is continuous and of bounded variation in the interval $(0, \infty)$. Although we shall need only the cosine transform of ${}_pF_{p+1}[-b^2x^2]$ $(b > 0)$, we record also in the next section the sine transform, since evidently both transforms are not readily available. Thus, we define for $a \geqslant 0$, $b > 0$

$$\mathscr{S}(a, b) \equiv \int_0^{\infty} \sin(2ax)\,{}_pF_{p+1}[(\alpha_p);\ (\beta_{p+1});\ -b^2x^2]\,\mathrm{d}x, \tag{1.3a}$$

$$\mathscr{C}(a, b) \equiv \int_0^{\infty} \cos(2ax)\,{}_pF_{p+1}[(\alpha_p);\ (\beta_{p+1});\ -b^2x^2]\,\mathrm{d}x. \tag{1.3b}$$

For conciseness, in what follows, we write for positive integers p

$$\Gamma((\alpha_p)) \equiv \Gamma(\alpha_1)\Gamma(\alpha_2) \cdots \Gamma(\alpha_p),$$

where if the sequence (α_p) is empty, the product of gamma functions reduces to unity. An asterisk (*) is used only in conjunction with a k-summation so that $(\alpha_p)^*$ means that α_k is not included in the sequence (α_p) for the current value of the summation index k; thus,

$$\Gamma((\alpha_p)^*) = \Gamma(\alpha_1) \cdots \Gamma(\alpha_{k-1})\Gamma(\alpha_{k+1}) \cdots \Gamma(\alpha_p).$$

We adapt throughout the convention that when the upper limit of a summation is less than the initial value of the lower limit, then the summation vanishes.

2. The sine and cosine transforms of ${}_pF_{p+1}[-b^2x^2]$

Miller has shown that the Mellin transform

$$F_p(s) \equiv \int_0^{\infty} x^{s-1}\,{}_0F_1[-;\ 1+\mu;\ -a^2x^2]\,{}_pF_{p+1}[(\alpha_p);\ (\beta_{p+1});\ -b^2x^2]\,\mathrm{d}x \tag{2.1a}$$

is given by [3, Eqs. (4.4)]

$$F_p(s) = \frac{1}{2} a^{-s} \frac{\Gamma(s/2)\Gamma(1+\mu)}{\Gamma(1+\mu-s/2)}\,{}_{p+2}F_{p+1}\left[\begin{matrix}(\alpha_p),\ s/2,\ s/2-\mu; \\ (\beta_{p+1});\end{matrix}\ \frac{b^2}{a^2}\right] \quad (0 < b < a) \tag{2.1b}$$

and

$$F_p(s) = \frac{1}{2} b^{-s} \frac{\Gamma(s/2)\Gamma((\alpha_p)-s/2)\Gamma((\beta_{p+1}))}{\Gamma((\alpha_p))\Gamma((\beta_{p+1})-s/2)}\,{}_{p+2}F_{p+1}\left[\begin{matrix}s/2,\ 1+s/2-(\beta_{p+1}); \\ 1+s/2-(\alpha_p),\ 1+\mu;\end{matrix}\ \frac{a^2}{b^2}\right]$$
$$+ \frac{1}{2} a^{-s} \sum_{k=1}^{p} \left(\frac{a^2}{b^2}\right)^{\alpha_k} \frac{\Gamma(1+\mu)\Gamma((\beta_{p+1}))\Gamma(s/2-\alpha_k)\Gamma((\alpha_p)^*-\alpha_k)}{\Gamma((\beta_{p+1})-\alpha_k)\Gamma(1+\mu+\alpha_k-s/2)\Gamma((\alpha_p)^*)}$$

$$\times {}_{p+2}F_{p+1}\left[\begin{matrix}\alpha_k,\ 1+\alpha_k-(\beta_{p+1});\\ 1+\alpha_k-s/2,\ 1+\mu+\alpha_k-s/2,\ 1+\alpha_k-(\alpha_p)^*;\end{matrix}\ \frac{a^2}{b^2}\right]\quad (0<a<b), \tag{2.1c}$$

where, for convergence of the integral defining $F_p(s)$,

$$0<\mathrm{Re}(s)<\mathrm{Re}(\tfrac{3}{2}+\mu+2\alpha_k)\quad (k=1,2,\ldots,p)$$

and

$$0<\mathrm{Re}(s)<\mathrm{Re}\left(1+\mu+\sum_{k=1}^{p+1}\beta_k-\sum_{k=1}^{p}\alpha_k\right).$$

It is understood that when $p=0$, the k-summation in Eq. (2.1c) is empty and the penultimate conditional inequality is superfluous so that Eqs. (2.1) reduce to the hypergeometric formulation of the discontinuous integral of Weber and Schafheitlin (cf. [5, Eqs. (1.2a) and (1.2c); 4, Eqs. (2.3) and (2.4)]). When $p=1$, Eqs. (2.1) reduce to the result that was deduced previously in [5, Eqs. (5.1)]. In what follows, and in particular, in Eqs. (1.1) and (2.1), it is tacitly assumed that the complex parameters are such that the expressions make sense.

In Eqs. (2.1) if we let $s=1$, $\mu=-\frac{1}{2}$ and note that $\cos z={}_0F_1[-;1/2;-z^2/4]$, then recalling the definition for the cosine transform given by Eq. (1.3b) we have the following lemma.

Lemma 1.

$$\mathscr{C}(a,b)=0\quad (0<b<a) \tag{2.2a}$$

and

$$\begin{aligned}\mathscr{C}(a,b)=&\frac{\sqrt{\pi}}{2b}\frac{\Gamma((\alpha_p)-\frac{1}{2})}{\Gamma((\beta_{p+1})-\frac{1}{2})}\frac{\Gamma((\beta_{p+1}))}{\Gamma((\alpha_p))}\,{}_{p+1}F_p\left[\begin{matrix}\frac{3}{2}-(\beta_{p+1});\\ \frac{3}{2}-(\alpha_p);\end{matrix}\ \frac{a^2}{b^2}\right]\\ &+\frac{\sqrt{\pi}}{2a}\frac{\Gamma((\beta_{p+1}))}{\Gamma((\alpha_p))}\sum_{k=1}^{p}\frac{\Gamma(\frac{1}{2}-\alpha_k)\Gamma((\alpha_p)^*-\alpha_k)}{\Gamma((\beta_{p+1})-\alpha_k)}\left(\frac{a^2}{b^2}\right)^{\alpha_k}\\ &\times {}_{p+1}F_p\left[\begin{matrix}1+\alpha_k-(\beta_{p+1});\\ \frac{1}{2}+\alpha_k,\ 1+\alpha_k-(\alpha_p)^*;\end{matrix}\ \frac{a^2}{b^2}\right]\quad (0<a<b),\end{aligned} \tag{2.2b}$$

where for convergence

$$\mathrm{Re}(\alpha_k)>0\quad (k=1,2,\ldots,p) \tag{2.2c}$$

and

$$\mathrm{Re}\left(\sum_{k=1}^{p+1}\beta_k-\sum_{k=1}^{p}\alpha_k\right)>\frac{1}{2}. \tag{2.2d}$$

By using an asymptotic result for ${}_pF_{p+1}[-b^2x^2]$ (see Eq. (3.1) below) and employing Chartier's test [8, p. 72] for infinite integrals involving periodic functions, it is not difficult to see that the integral $\mathscr{C}(0, b)$ converges provided that the conditional inequality (2.2d) holds true and $\mathrm{Re}(\alpha_k) > \frac{1}{2}$ for $k = 1, 2, \ldots, p$. Thus, letting $a \to 0$ in Eq. (2.2b) we have the following lemma.

Lemma 2. *For $b > 0$*

$$\mathscr{C}(0, b) = \frac{\sqrt{\pi}}{2b} \frac{\Gamma((\beta_{p+1}))}{\Gamma((\alpha_p))} \frac{\Gamma((\alpha_p) - \frac{1}{2})}{\Gamma((\beta_{p+1}) - \frac{1}{2})}, \tag{2.3}$$

where for convergence

$$\mathrm{Re}(\alpha_k) > \tfrac{1}{2} \quad (k = 1, 2, \ldots, p)$$

and

$$\mathrm{Re}\left(\sum_{k=1}^{p+1} \beta_k - \sum_{k=1}^{p} \alpha_k\right) > \frac{1}{2}.$$

This result may also be obtained from a table of Mellin transforms (see e.g. [6, Section 8.4.51(1)]).

Now setting $s = 2$, $\mu = \frac{1}{2}$ in Eqs. (2.1), noting that $\sin z = z\,{}_0F_1[-; 3/2; -z^2/4]$, and recalling the definition of the sine transform given by Eq. (1.3a) we have

$$\mathscr{S}(a, b) = \frac{1}{2a}\, {}_{p+2}F_{p+1}\left[\begin{matrix} 1, \frac{1}{2}, (\alpha_p); & \dfrac{b^2}{a^2} \\ (\beta_{p+1}); & \end{matrix}\right] \quad (0 < b < a)$$

and

$$\begin{aligned} \mathscr{S}(a, b) = {} & \frac{a \prod_{k=1}^{p+1} (\beta_k - 1)}{b^2 \prod_{k=1}^{p} (\alpha_k - 1)}\, {}_{p+2}F_{p+1}\left[\begin{matrix} 1, 2 - (\beta_{p+1}); & \dfrac{a^2}{b^2} \\ \frac{3}{2}, 2 - (\alpha_p); & \end{matrix}\right] \\ & + \frac{\sqrt{\pi}}{2a} \sum_{k=1}^{p} \frac{\Gamma((\beta_{p+1}))\Gamma(1 - \alpha_k)}{\Gamma((\alpha_p)^*)\Gamma(\frac{1}{2} + \alpha_k)} \frac{\Gamma((\alpha_p)^* - \alpha_k)}{\Gamma((\beta_{p+1}) - \alpha_k)} \left(\frac{a^2}{b^2}\right)^{\alpha_k} \\ & \times {}_{p+1}F_p\left[\begin{matrix} 1 + \alpha_k - (\beta_{p+1}); & \dfrac{a^2}{b^2} \\ \frac{1}{2} + \alpha_k, 1 + \alpha_k - (\alpha_p)^*; & \end{matrix}\right] \quad (0 < a < b), \end{aligned}$$

where for convergence the conditional inequalities (2.2c) and (2.2d) hold true.

3. Evaluation of the hypergeometric sums

Since for $|z| \to \infty$, $|\arg(z)| < \pi/2$

$$\begin{aligned} {}_pF_{p+1}[(\alpha_p); (\beta_{p+1}); -z^2] \sim {} & \sum_{k=1}^{p} A_k \left(\frac{1}{z^2}\right)^{\alpha_k} \\ & + A_{p+1} \left(\frac{1}{z}\right)^{-\frac{1}{2} + \sum_{k=1}^{p+1} \beta_k - \sum_{k=1}^{p} \alpha_k} \cos(2z + A_{p+2}), \end{aligned} \tag{3.1}$$

where the $A_k (k = 1,2, \ldots, p+2)$ are dependent of the parameters of ${}_pF_{p+1}[-z^2]$ (see e.g. [3, Eq. (2.2b)]), it is not difficult to infer that the sum $S(x)$ defined by Eq. (1.1a) converges conditionally (see e.g. [1, pp. 315–316]) provided that

$$\mathrm{Re}(\alpha_k) > 0 \quad (k = 1,2, \ldots, p)$$

and

$$\mathrm{Re}\left(\sum_{k=1}^{p+1} \beta_k - \sum_{k=1}^{p} \alpha_k\right) > \frac{1}{2}.$$

In a similar way, we see that the sum $T(x)$ defined by Eq. (1.1b) converges conditionally provided that the latter inequality holds true and

$$\mathrm{Re}(\alpha_k) > \tfrac{1}{2} \quad (k = 1, 2, \ldots, p).$$

Now, applying Poisson's formula given by Eq. (1.2) to the hypergeometric sums defined by Eqs. (1.1), and utilizing the notation for the cosine transform introduced in Eq. (1.3b), we find, as in [2] mutatus mutandis, for $x > 0$

$$S(x) = -\frac{1}{2} + 2\sum_{k=1}^{\infty} \mathscr{C}\left(\pi\left(k - \frac{1}{2}\right), x\right) \tag{3.2a}$$

and

$$T(x) = -\frac{1}{2} + \mathscr{C}(0, x) + 2\sum_{k=1}^{\infty} \mathscr{C}(\pi k, x). \tag{3.2b}$$

Finally, by using, respectively, Lemma 1 together with Eq. (3.2a), and Lemmas 1 and 2 together with Eq. (3.2b), we deduce the following:

Theorem. *For* $x > 0$

$$\begin{aligned}\sum_{n=1}^{\infty} (-1)^n {}_pF_{p+1}[(\alpha_p); (\beta_{p+1}); -n^2x^2] &= -\frac{1}{2} \\ &+ \frac{\sqrt{\pi}}{x}\frac{\Gamma((\beta_{p+1}))}{\Gamma((\alpha_p))}\frac{\Gamma((\alpha_p)-\frac{1}{2})}{\Gamma((\beta_{p+1})-\frac{1}{2})}\sum_{\ell=1}^{\sigma} {}_{p+1}F_p\left[\begin{matrix}\frac{3}{2}-(\beta_{p+1}); \\ \frac{3}{2}-(\alpha_p);\end{matrix}\ \frac{\pi^2}{x^2}\left(\ell-\frac{1}{2}\right)^2\right] \\ &+ \frac{1}{\sqrt{\pi}}\frac{\Gamma((\beta_{p+1}))}{\Gamma((\alpha_p))}\sum_{k=1}^{p}\frac{\Gamma((\alpha_p)^*-\alpha_k)\Gamma(\frac{1}{2}-\alpha_k)}{\Gamma((\beta_{p+1})-\alpha_k)}\left(\frac{\pi^2}{x^2}\right)^{\alpha_k} \\ &\times \sum_{\ell=1}^{\sigma}\left(\ell-\frac{1}{2}\right)^{2\alpha_k-1} {}_{p+1}F_p\left[\begin{matrix}1+\alpha_k-(\beta_{p+1}); \\ \frac{1}{2}+\alpha_k,\ 1+\alpha_k-(\alpha_p)^*;\end{matrix}\ \frac{\pi^2}{x^2}\left(\ell-\frac{1}{2}\right)^2\right],\end{aligned} \tag{3.3a}$$

where σ *is a nonnegative integer such that* $(\sigma - \frac{1}{2})\pi < x < (\sigma + \frac{1}{2})\pi$ *and for convergence of the series*

$$\mathrm{Re}(\alpha_k) > 0 \quad (k = 1,2, \ldots, p), \tag{3.3b}$$

$$\mathrm{Re}\left(\sum_{k=1}^{p+1} \beta_k - \sum_{k=1}^{p} \alpha_k\right) > \frac{1}{2}; \tag{3.3c}$$

and

$$\sum_{n=1}^{\infty} {}_pF_{p+1}[(\alpha_p); (\beta_{p+1}); \; -n^2x^2] = -\frac{1}{2}$$
$$+\frac{\sqrt{\pi}}{x}\frac{\Gamma((\beta_{p+1}))}{\Gamma((\alpha_p))}\frac{\Gamma((\alpha_p)-\frac{1}{2})}{\Gamma((\beta_{p+1})-\frac{1}{2})}\left(\frac{1}{2}+\sum_{\ell=1}^{\sigma} {}_{p+1}F_p\left[\begin{matrix}\frac{3}{2}-(\beta_{p+1}); & \frac{\pi^2\ell^2}{x^2}\\ \frac{3}{2}-(\alpha_p); & \end{matrix}\right]\right)$$
$$+\frac{1}{\sqrt{\pi}}\frac{\Gamma((\beta_{p+1}))}{\Gamma((\alpha_p))}\sum_{k=1}^{p}\frac{\Gamma((\alpha_p)^*-\alpha_k)\Gamma(\frac{1}{2}-\alpha_k)}{\Gamma((\beta_{p+1})-\alpha_k)}\left(\frac{\pi^2}{x^2}\right)^{\alpha_k}$$
$$\times\sum_{\ell=1}^{\sigma}\ell^{2\alpha_k-1}{}_{p+1}F_p\left[\begin{matrix}1+\alpha_k-(\beta_{p+1}); & \frac{\pi^2\ell^2}{x^2}\\ \frac{1}{2}+\alpha_k,\; 1+\alpha_k-(\alpha_p)^*; & \end{matrix}\right], \tag{3.4a}$$

where σ is a nonnegative integer such that $\pi\sigma < x < \pi(\sigma+1)$ and for convergence of the series

$$\mathrm{Re}(\alpha_k) > \tfrac{1}{2} \quad (k = 1, 2, \dots, p), \tag{3.4b}$$

$$\mathrm{Re}\left(\sum_{k=1}^{p+1}\beta_k - \sum_{k=1}^{p}\alpha_k\right) > \frac{1}{2}. \tag{3.4c}$$

If $\sigma = 0$ in Eqs. (3.3a) and (3.4a), respectively, then we have the

Corollary. *For* $0 < x < \pi/2$

$$\sum_{n=1}^{\infty}(-1)^n{}_pF_{p+1}[(\alpha_p); (\beta_{p+1}); \; -n^2x^2] = -\frac{1}{2}, \tag{3.5a}$$

where the inequalities (3.3b) *and* (3.3c) *hold true; and for* $0 < x < \pi$

$$\sum_{n=1}^{\infty} {}_pF_{p+1}[(\alpha_p); (\beta_{p+1}); \; -n^2x^2] = -\frac{1}{2}+\frac{\sqrt{\pi}}{2x}\frac{\Gamma((\beta_{p+1}))}{\Gamma((\alpha_p))}\frac{\Gamma((\alpha_p)-\frac{1}{2})}{\Gamma((\beta_{p+1})-\frac{1}{2})}, \tag{3.5b}$$

where the inequalities (3.4b) *and* (3.4c) *hold true.*

Eq. (3.5a) is valid in the interval $-\frac{1}{2}\pi < x < \frac{1}{2}\pi$, since both sides of the result are even functions of x and the left-hand side is Cesàro summable at $x = 0$. In the case $p = 0$, Eqs. (3.3) reduces to a result due to Nielsen (ca 1900), and Eqs. (3.4) reduce to a result apparently due to Titchmarsh (ca 1937). The previous theorem specialized with $p = 1$ is recorded in [2], where other pertinent references are given.

References

[1] K. Knopp, Theory and Application of Infinite Series, Blackie, London, 1951.
[2] A.R. Miller, On certain Schlömilch-type series, J. Comput. Appl. Math. 80 (1997) 83–95.

[3] A.R. Miller, On the Mellin transform of products of Bessel and generalized hypergeometric functions, J. Comput. Appl. Math., to appear.
[4] A.R. Miller, H. Exton, Sonine–Gegenbauer-type integrals, J. Comput. Appl. Math. 55 (1994) 289–310.
[5] A.R. Miller, H.M. Srivastava, On the Mellin transform of a product of hypergeometric functions, J. Austral. Math. Soc. Series B, to appear.
[6] A.P. Prudnikov, Yu.A. Brychkov, O.I. Marichev, Integrals and Series, vol. 3, Gordon and Breach, New York, 1990.
[7] E.C. Titchmarsh, Introduction to the Theory of Fourier Integrals, 2nd ed., Clarendon Press, Oxford, 1948.
[8] E.T. Whittaker, G.N. Watson, A Course of Modern Analysis, 4th ed., Cambridge University Press, Cambridge, 1927.